

INEQUALITY ON $t_\nu(K)$ DEFINED BY LIVINGSTON AND NAIK AND ITS APPLICATIONS

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ABSTRACT. Let $D_+(K, t)$ denote the positive t -twisted double of K . For a fixed integer-valued additive concordance invariant ν that bounds the smooth four genus of a knot and determines the smooth four genus of positive torus knots, Livingston and Naik defined $t_\nu(K)$ to be the greatest integer t such that $\nu(D_+(K, t)) = 1$. Let K_1 and K_2 be any knots then we prove the following inequality : $t_\nu(K_1) + t_\nu(K_2) \leq t_\nu(K_1 \# K_2) \leq \min(t_\nu(K_1) - t_\nu(-K_2), t_\nu(K_2) - t_\nu(-K_1))$. As an application we show that $t_\tau(K) \neq t_s(K)$ for infinitely many knots and that their difference can be arbitrarily large, where $t_\tau(K)$ (respectively $t_s(K)$) is $t_\nu(K)$ when ν is Ozv th-Szab  invariant τ (respectively when ν is normalized Rasmussen s invariant).

1. INTRODUCTION

Let ν be any integer-valued concordance invariant with the following properties:

- (1) additive under connected sum,
- (2) $|\nu(K)| \leq g_4(K)$,
- (3) $\nu(T_{p,q}) = (p-1)(q-1)/2$ for $p, q > 0$.

Notice that the Ozv th-Szab  invariant τ satisfies the above properties [OS03], as does the Rasmussen s invariant when suitably normalized (i.e. when $\nu = -s/2$) [Ras10]. Let $D_\pm(K, t)$ denote the positive or negative t -twisted double of K . Then for a fixed concordance invariant ν , Livingston and Naik [LN06] show that $\nu(D_+(K, t))$ is always 1 or 0 (see Theorem 2.1) and define $t_\nu(K)$ to be the greatest integer t such that $\nu(D_+(K, t)) = 1$. Specializing to τ and s , we have the following two concordance invariants $t_\tau(K)$ (respectively $t_s(K)$) which is the greatest integer t where $\tau(D_+(K, t)) = 1$ (respectively $-s(D_+(K, t))/2 = 1$). Hedden and Ording [HO08] show that there exist K for which $t_\tau(K) \neq t_s(K)$, in particular they show that $t_\tau(T_{2,2n+1}) = 2n - 1$ whereas $t_s(T_{2,3}) \geq 2$, $t_s(T_{2,5}) \geq 5$, and $t_s(T_{2,7}) \geq 8$ (In fact, it is easy to verify that $t_s(T_{2,3}) = 2$ and $t_s(T_{2,5}) = 5$ using Bar-Natan's program [BN]). This was the first example known of a knot K for which $\tau(K) \neq -s(K)/2$. (Note that it is proven that $\tau \neq -s/2$ even for topologically slice knots by Livingston, [Liv08].) Further they make a remark that it would be reasonable to guess that $t_s(T_{2,2n+1}) = 3n - 1$, which would imply that $t_\tau(K) \neq t_s(K)$ for infinitely many different knots. Also, Hedden [Hed07] showed that $t_\tau(K)$ does not give more information than $\tau(K)$:

Theorem 1.1. [Hed07, Theorem 1.5] $t_\tau(K) = 2\tau(K) - 1$.

However, $t_s(K)$ is not well understood. In this paper we show the following inequality :

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Theorem 1.2. *Let K_1 and K_2 be any knots and ν be any integer-valued concordance invariant with properties (1), (2), and (3) as above then the following inequality hold:*

$$t_\nu(K_1) + t_\nu(K_2) \leq t_\nu(K_1 \# K_2) \leq \min(t_\nu(K_1) - t_\nu(-K_2), t_\nu(K_2) - t_\nu(-K_1)).$$

We have the following as the immediate corollary:

Corollary 1.3. *For any positive integer n , there exists a knot K_n such that*

$$|t_\tau(K_n) - t_s(K_n)| > n.$$

Proof. Let K_n be n connected sum of $T_{2,5}$. Then by Theorem 1.2 and the fact that $t_\tau(K_n) = n \cdot \tau(T_{2,5}) - 1 = 4n - 1$ and $t_s(T_{2,5}) \geq 5$ by Theorem 1.1 and [HO08], the result follows. \square

We end this section with the following remark:

Remark 1.4. If we assume that $t_s(K)$ is a polynomial of $-s(K)/2$ with integer coefficients, it is easy to verify that $t_s(K) = 3 \cdot (-s(K)/2) - 1$ using Theorem 1.2. Then in the light of [HO08] it would be reasonable to guess that $t_s(K) = 3 \cdot (-s(K)/2) - 1$.

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2. PROOF OF THE THEOREM 1.2

We will denote by $TB(K)$ the maximum value of the Thurston-Bennequin number, taken over all possible Legendrian representatives of K . Recall the following theorem from [LN06]:

Theorem 2.1. [LN06, Theorem 2] *For each knot K there is an integer $t_\nu(K)$ such that:*

$$\nu(D_+(K, t)) = \begin{cases} 0 & \text{for } t > t_\nu(K) \\ 1 & \text{for } t \leq t_\nu(K), \end{cases}$$

where $t_\nu(K)$ satisfies $TB(K) \leq t_\nu(K) < -TB(-K)$.

A similar result holds for $D_-(K, t)$ using $t_\nu(-K)$:

$$\nu(D_-(K, t)) = \begin{cases} -1 & \text{for } t \geq -t_\nu(-K) \\ 0 & \text{for } t < -t_\nu(-K), \end{cases}$$

where $t_\nu(-K)$ satisfies $TB(-K) \leq t_\nu(-K) < -TB(K)$.

Now, we are ready to prove the Theorem 1.2. The proof completely relies on Theorem 2.1.

Proof of Theorem 1.2. Let t_1 and t_2 be integers and consider $D_+(K_1, t_1) \# D_+(K_2, t_2)$ and $D_+(K_1 \# K_2, t_1 + t_2)$. Then there is a genus one cobordism from $D_+(K_1, t_1) \# D_+(K_2, t_2)$ to $D_+(K_1 \# K_2, t_1 + t_2)$ (see Figure 1). Hence if $\nu(D_+(K_1, t_1)) = \nu(D_+(K_2, t_2)) = 1$, then $\nu(D_+(K_1 \# K_2, t_1 + t_2)) = 1$. Letting $t_1 = t_\nu(K_1)$ and $t_2 = t_\nu(K_2)$, we have $\nu(D_+(K_1, t_1)) = \nu(D_+(K_2, t_2)) = 1$ by Theorem 2.1. Using Theorem 2.1 again we have $t_\nu(K_1) + t_\nu(K_2) \leq t_\nu(K_1 \# K_2)$.

Using a similar argument, notice that there is a genus one cobordism from $D_+(K_1, t_1) \# D_-(K_2, t_2)$ to $D_+(K_1 \# K_2, t_1 + t_2)$ by simply changing the sign of the clasp in Figure 1. Therefore if $\nu(D_+(K_1, t_1)) = 0$ and $\nu(D_-(K_2, t_2)) = -1$, then $\nu(D_+(K_1 \# K_2, t_1 + t_2)) = 0$ by Theorem 2.1. Letting $t_1 = t_\nu(K_1) + 1$ and $t_2 = -t_\nu(-K_2)$, we have $\nu(D_+(K_1, t_1)) = 0$ and $\nu(D_-(K_2, t_2)) = -1$ by Theorem 2.1. Using Theorem 2.1 again we have $t_\nu(K_1) + 1 - t_\nu(-K_2) \geq t_\nu(K_1 \# K_2) + 1$, hence $t_\nu(K_1 \# K_2) \leq t_\nu(K_1) - t_\nu(-K_2)$. Finally, by switching roles of K_1 and K_2 we also get $t_\nu(K_1 \# K_2) \leq t_\nu(K_2) - t_\nu(-K_1)$ which completes the proof. \square

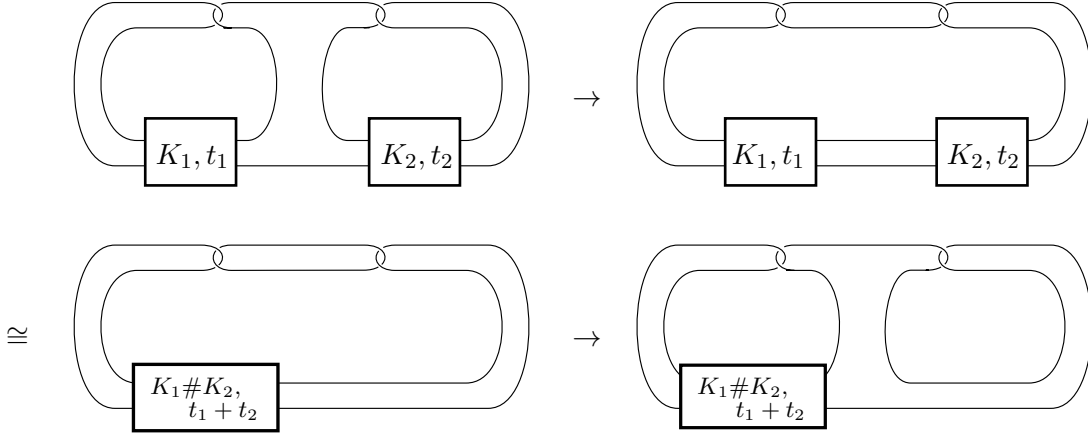


FIGURE 1. A genus one cobordism from $D_+(K_1, t_1) \# D_+(K_2, t_2)$ to $D_+(K_1 \# K_2, t_1 + t_2)$. The top left figure is $D_+(K_1, t_1) \# D_+(K_2, t_2)$, the top right figure is obtained from the top left figure after one band sum, the bottom left figure is obtained from the top right figure after an isotopy, and the bottom right figure is obtained from the bottom left figure after one band sum and it is isotopic to $D_+(K_1 \# K_2, t_1 + t_2)$.

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